

A MONODROMY GRAPH APPROACH TO THE PIECEWISE POLYNOMIALITY OF MIXED DOUBLE HURWITZ NUMBERS

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ABSTRACT. Hurwitz numbers count genus g , degree d covers of the complex projective line with fixed branched locus and fixed ramification data. An equivalent description is given by factorisations in the symmetric group. Double Hurwitz numbers are a class of Hurwitz-type counts of specific interest. In recent years a related counting problem in the context of random matrix theory was introduced as so-called monotone Hurwitz numbers. These can be viewed as a desymmetrised version of the Hurwitz-problem. A combinatorial interpolation between classical and monotone double Hurwitz numbers was introduced as mixed double Hurwitz numbers and it was proved that these objects are piecewise polynomial in a certain sense. The aim of this paper is twofold: Using a connection between mixed double Hurwitz numbers and tropical covers in terms of so-called monodromy graphs, we give algorithms to compute the polynomials for mixed Hurwitz numbers in all genera using Erhart theory. We further use this approach to study the wall-crossing behaviour of mixed Hurwitz numbers in genus 0 in terms of related Hurwitz-type counts.

1. INTRODUCTION

Hurwitz numbers are important enumerative invariants connecting various areas of mathematics, such as algebraic geometry, combinatorics, representation theory, operator theory, tropical geometry and many more. Introduced by Adolf Hurwitz in 1891, they were used to study the moduli space of genus g curves [Hur91]. There are several equivalent definitions of Hurwitz numbers. The one originally introduced by Hurwitz is of topological nature: It counts the number of branched genus g , degree d covers of \mathbb{P}^1 with fixed ramification data over n fixed points. Another description, which is essentially due to Hurwitz as well, interprets Hurwitz numbers as the enumeration of certain factorisations in the symmetric group. Hurwitz numbers have been a focal point of study in the last two decades when their relationship to Gromov-Witten theory and mathematical physics was uncovered. The cases of single and double Hurwitz numbers have proven to be of specific interest. Single Hurwitz numbers count covers of \mathbb{P}^1 with ramification profile μ at $0 \in \mathbb{P}^1$ (where μ is a partition of the degree) and simple ramification data at m arbitrary fixed points (where m is determined by the Riemann-Hurwitz formula). Double Hurwitz numbers count covers of \mathbb{P}^1 with ramification profile μ at 0 and ν at ∞ and simple ramification at m arbitrary fixed points (where again m is given by the Riemann Hurwitz formula). A remarkable result relating single Hurwitz numbers to intersection products is the celebrated ELSV formula [ELSV01]. An immediate consequence is that — up to a combinatorial factor — single Hurwitz numbers behave polynomially in the entries of the partition μ specifying the ramification data over 0 . Many properties of single Hurwitz numbers translate to similar properties of double Hurwitz numbers, e.g. it was proved in [GJV05] that double Hurwitz numbers behave piecewise polynomially in the entries of the partitions μ and ν specifying the ramification data over

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0 and ∞ . The natural question whether there is an ELSV-type formula for double Hurwitz numbers remains an active field of research. ELSV-type formulas are closely related to the Chekhov-Eynard-Orantin topological recursion ([CE06], [EMS11]), which is a way of associating a recursion involving differential forms to a spectral curve. There are spectral curves with which differential forms can be produced that can be viewed as generating functions for single Hurwitz numbers — one can say that single Hurwitz numbers satisfy the topological recursion. The question whether double Hurwitz numbers satisfy the topological recursion in some sense is also an active field of research ([ACEH16]). There have been several cases where an ELSV formula was derived from topological recursion ([DLPS15], [ALS16]). By proving that an enumerative problem satisfies the topological recursion, one often makes use of the (quasi-)polynomiality of this problem. In connection with topological recursion and ELSV-formulas, the piecewise polynomial structure of double Hurwitz numbers gains relevance.

The chamber behaviour induced by the piecewise polynomial structure of double Hurwitz numbers was first studied in [SSV08] and by using degeneration techniques, wall-crossing formulas for genus 0 were given. The problem of understanding the chamber behaviour for higher genera remained unanswered until wall-crossing formulas for arbitrary genus were given in [CJM11] and [Joh15]. In [CJM11], these formulas were proved using so-called monodromy graphs which essentially express double Hurwitz numbers in terms of covers as they appear in tropical geometry. This description was developed in [CJM10] by giving a graph theoretic interpretation of factorisations in the symmetric group.

A different class of Hurwitz-type counts was introduced in [GGPN14] as so-called *monotone Hurwitz numbers*. They are defined in the symmetric group setting and show up in the computation of the HCIZ integral. A tropical interpretation of certain monotone Hurwitz numbers in the flavour of [CJM10] was developed in [DK16], where a conjecture on the topological recursion on single monotone Hurwitz numbers was stated (for further literature on monotone Hurwitz numbers and topological recursion, see e.g. [ALS16], [DDM14]).

A combinatorial interpolation between double Hurwitz numbers and monotone double Hurwitz numbers was studied in [GGN16] as *mixed Hurwitz numbers*. It was proved that those new enumerative objects satisfy a piecewise polynomiality result as well, thus proving piecewise polynomiality for monotone double Hurwitz numbers and giving a new proof for the piecewise polynomiality of double Hurwitz numbers. This was done in terms of character theory.

It is natural to ask for wall-crossing formulas for mixed Hurwitz numbers. In this paper we combine several approaches to Hurwitz numbers to answer this question. We begin by introducing a tropical interpretation of mixed Hurwitz numbers in the flavour of [DK16]. Using this description, we develop an algorithm to compute the polynomials in each chamber (see Algorithm 4.9 for the case of genus 0 and Algorithm 4.17 for the case of arbitrary genus, which involves Erhart theory, more precisely the integration over lattice points in a polytope. Moreover, we generalise this method to give an algorithm computing the polynomials in each chamber by certain polytope-computations for arbitrary genus.

Finally, we introduce a Hurwitz-type counting problem generalising mixed Hurwitz numbers in genus 0, which is accessible by our algorithms as well. We give recursive wall-crossing formulas for this generalised counting problem, which in particular implies wall-crossing formulas for mixed Hurwitz numbers.

In section 2, we recall some of the basic facts about Hurwitz numbers and outline the previous polynomiality results. In section 3, we describe mixed Hurwitz numbers in terms of monodromy graphs. We use this description in section 4 to compute the polynomials in each chamber. We begin this discussion for genus 0 in section 4.1 and generalise the method for

higher genus in section 4.2. In section 5, we define a generalisation of mixed Hurwitz numbers in genus 0 and explain how this new enumerative problem yields a recursive wall-crossing formula, which is in particular valid for mixed Hurwitz numbers.

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2. PRELIMINARIES

We begin by introducing the basic notions of Hurwitz numbers.

Definition 2.1 (Double Hurwitz numbers). Let d be a positive integers, μ, ν two ordered partitions of d and let g be a non-negative integer. Moreover, let q_1, \dots, q_m be points in \mathbb{P}^1 , where $m = 2g - 2 + \ell(\mu) + \ell(\nu)$. We define a Hurwitz cover of type (g, μ, ν) to be a map $\pi : C \rightarrow \mathbb{P}^1$, such that:

- (1) C is a genus g curve,
- (2) π is a degree d map, with ramification profile μ over 0, ν over ∞ and $(2, 1, \dots, 1)$ over q_i for all $i = 1, \dots, m$,
- (3) π is unramified everywhere else,
- (4) the pre-images of 0 and ∞ are labeled, such that the point labeled i in $\pi^{-1}(0)$ (respectively $\pi^{-1}(\infty)$) has ramification index μ_i (respectively ν_i).

We define an isomorphism between two covers $\pi : C_1 \rightarrow \mathbb{P}^1$ and $\pi' : C_2 \rightarrow \mathbb{P}^1$ to be a homeomorphism $\phi : C_1 \rightarrow C_2$ respecting the labels, such that the following diagram commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{\phi} & C_2 \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{P}^1 & \xrightarrow{\text{id}} & \mathbb{P}^1 \end{array}$$

Then we define the double Hurwitz numbers as follows:

$$H_g(\mu, \nu) = \sum \frac{1}{|\text{Aut}(\pi)|},$$

where the sum goes over all isomorphism classes of Hurwitz covers of type (g, μ, ν) . This number does not depend on the position of the q_i . The degree is implicit in the notation $H_g(\mu, \nu)$, as $d = \sum \mu_i = \sum \nu_j$. The number m of simple branch points is determined by the Riemann-Hurwitz formula, so $m = 2g - 2 + \ell(\mu) + \ell(\nu)$ as above.

For $\sigma \in S_d$, we denote its cycle type by $\mathcal{C}(\sigma) \vdash d$. We define the following factorisation counting problem in the symmetric group:

Definition 2.2 (Factorisation the symmetric group). Let d, g, μ, ν be as in Definition 2.1. We call $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ a factorisation of type (g, μ, ν) , if:

- (1) $\sigma_1, \sigma_2, \tau_i \in \mathcal{S}_d$,
- (2) $\sigma_2 \cdot \tau_m \cdot \dots \cdot \tau_1 \cdot \sigma_1 = \text{id}$,
- (3) $m = 2g - 2 + \ell(\mu) + \ell(\nu)$
- (4) $\mathcal{C}(\sigma_1) = \mu$, $\mathcal{C}(\sigma_2) = \nu$ and $\mathcal{C}(\tau_i) = (2, 1, \dots, 1)$,
- (5) the group generated by $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ acts transitively on $\{1, \dots, d\}$,

(6) the disjoint cycles of σ_1 and σ_2 are labeled, such that the cycle i has length μ_i . We denote the set of all factorisations of type (g, μ, ν) by $\mathcal{F}(g, \mu, \nu)$.

A well-known fact is the following theorem, which is essentially due to Hurwitz.

Theorem 2.3. *Let g, μ, ν as in the previous definition, then*

$$H_g(\mu, \nu) = \frac{1}{d!} |\mathcal{F}(g, \mu, \nu)|.$$

As proved in [GGPN14] monotone double Hurwitz numbers appear as the coefficients of the HCIZ-integral. They can be defined as counts of factorisations as in Definition 2.2 by imposing an additional condition on the transpositions:

Definition 2.4 (Monotone double Hurwitz numbers). Let k be a non-negative integer. We call $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ a *monotone factorisation* of type (g, μ, ν) , if it is a factorisation of type (g, μ, ν) , such that:

(7a) If $\tau_i = (r_i \ s_i)$ with $r_i < s_i$, we have $s_i \leq s_{i+1}$ for all $i = 1, \dots, m-1$.

Let $\vec{\mathcal{F}}(g, \mu, \nu)$ be the set of all monotone factorisations of type (g, μ, ν) . Then we define the *monotone Hurwitz number* to be:

$$\vec{H}_g(\mu, \nu) = \frac{1}{d!} |\vec{\mathcal{F}}(g, \mu, \nu)|.$$

So far, there exists no equivalent definition similar to Definition 2.1 in terms of topological covers.

In [GGN16], a combinatorial interpolation between classical and monotone double Hurwitz numbers was introduced. The idea is to impose the monotonicity condition (7) only on the first k transpositions.

Definition 2.5 (Mixed Hurwitz numbers). Let d, g, μ, ν be as in Definition 2.1 and let k be a non-negative integer. We define a mixed factorisation of type (g, μ, ν, k) to be a factorisation $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ of type (g, μ, ν) satisfying the following additional condition:

(7b) If $\tau_i = (r_i \ s_i)$ with $r_i < s_i$, we have $s_i \leq s_{i+1}$ for all $i = 1, \dots, k-1$.

Let $\mathcal{F}(g, \mu, \nu, k)$ be the set of all mixed factorisations of type (g, μ, ν, k) . Then we define the *mixed Hurwitz number* to be:

$$H_g^k(\mu, \nu) = \frac{1}{d!} |\mathcal{F}(g, \mu, \nu, k)|.$$

Fixing the length μ and ν , we can view mixed Hurwitz numbers as a function

$$H_g^k : \mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)} \rightarrow \mathbb{Q} \\ (\mu, \nu) \mapsto H_g^k(\mu, \nu),$$

where $\ell(\mu)$ (resp. $\ell(\nu)$) is the length of μ (resp. ν). For each $I \subset \{1, \dots, \ell(\mu)\}, J \subset \{1, \dots, \ell(\nu)\}$ we obtain linear equations $\sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, where the μ_i (resp. ν_j) are the coordinates in $\mathbb{N}^{\ell(\mu)}$ (resp. $\mathbb{N}^{\ell(\nu)}$). The equations induce a hyperplane arrangement \mathcal{W} in $\mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)}$. By considering the complement on \mathcal{W} this hyperplane arrangement divides $\mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)}$ into chambers C .

Theorem 2.6 ([GJV05], [GGN16]). *The function H_g^k described above is piecewise polynomial, i.e. for each chamber C there exists a polynomial $h_g^k(C) \in \mathbb{Q}[\underline{M}, \underline{N}]$, where $\underline{M} = M_1, \dots, M_{\ell(\mu)}$ and $\underline{N} = N_1, \dots, N_{\ell(\nu)}$, such that $H_g^k(\mu, \nu) = h_g^k(C)(\mu, \nu)$.*

3. MIXED HURWITZ NUMBERS VIA MONODROMY GRAPHS

In [DK16], Lemma 7, the following result was proved for monotone factorisations. It may be easily adapted for mixed factorisations. For the convenience of the reader we give the proof. We work in the symmetric group ring. For an introduction to these techniques and their connection to Hurwitz numbers, see e.g. [CM16] chapter 9.

Lemma 3.1. *Fix a permutation σ of cycle type μ and a nonnegative number k . The number of mixed factorisation $(\sigma_1, \tau_1, \dots, \tau_r, \sigma_2)$ of type (g, μ, ν, k) , satisfying $\sigma_1 = \sigma$ does not depend on the choice of σ .*

Proof. Fix a permutation σ of cycle type μ . Let $K_{\tau,k}^\bullet(\sigma)$ be the number of mixed factorisations $(\sigma, \tau_1, \dots, \tau_m, \sigma_2)$ of type $\tau = (g, \mu, \nu, k)$ where we drop the transitivity condition. We can rewrite the equation as follows

$$\sigma_2^{-1} \tau_r \cdots \tau_1 = \sigma^{-1}.$$

We see that $K_{\tau,k}^\bullet(\sigma)$ is the coefficient of σ^{-1} in

$$(1) \quad C_\nu h_k(J_2, \dots, J_{|\nu|})(C_\kappa)^{r-k} \in \mathbb{C}[\mathcal{S}_d],$$

where $\kappa = (2, 1, \dots, 1) \vdash |\nu|$, C_w denotes the conjugacy class of permutations with cycle type w , h_i is the complete homogeneous symmetric function of degree i and J_i denote the Jucys-Murphy elements

$$J_i = (1, i) + \cdots + (i-1, i) \in \mathbb{C}[\mathcal{S}_d]$$

for $i = 2, \dots, |\nu|$. It is well known, that conjugacy classes and the Jucys-Murphy elements lie in the center of $\mathbb{C}[\mathcal{S}_d]$. Thus the expression in equation (1) is a linear combination of conjugacy classes and therefore all permutations in the same conjugacy class appear with the same coefficient. Thus $K_{\tau,k}^\bullet(\sigma)$ only depends on the conjugacy class of σ .

Now let $K_{\tau,k}^\circ(\sigma)$ be the number of factorisations as above that satisfy the transitivity condition. If σ is a d -cycle, where $d = \sum \mu_i$, then $K_{\tau,k}^\bullet(\sigma) = K_{\tau,k}^\circ(\sigma)$ and the result holds. For any permutation σ , set $\sigma = \Sigma_1 \cdots \Sigma_{\ell(\mu)}$ be the decomposition in disjoint cycles. We can decompose every non-transitive factorisation into a union of transitive factorisations. This leads to the following formula

$$K_{\tau,k}^\bullet(\sigma) = K_{\tau,k}^\circ(\sigma) + \sum_{s=2}^{\ell(\mu)} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_s = [k] \\ \mu^{(1)} \sqcup \cdots \sqcup \mu^{(s)} = \mu \\ k_1 + \cdots + k_s = k}} \prod_{l=1}^s K_{\tau,k}^\circ(\Sigma_{I_l}),$$

where the summation is over partitions of $[k] = \{1, \dots, k\}$ into disjoint non-empty subsets $I_1 \sqcup \cdots \sqcup I_s$, ordered tuples of partitions $\mu^{(1)} \sqcup \cdots \sqcup \mu^{(s)}$ whose union is μ . Moreover, for $I \subset [k]$, we let Σ_I denote the permutation obtained by taking the product of all cycles Σ_i for $i \in I$. We already proved that $K_{\tau,k}^\bullet(\sigma)$ only depends on the cycle type of σ , by induction on the length of μ the terms $K_{\tau,k}^\circ(\Sigma_{I_l})$ only depend on the cycle type of Σ_{I_l} , thus $K_{\tau,k}^\circ(\sigma)$ only depends on the cycle type of σ and we are finished. \square

Thus, in order to compute $\vec{H}_g^k(\mu, \nu)$, we do not have to count all mixed factorisations of type (g, μ, ν, k) , rather we may compute the number of mixed factorisations $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ of type (g, μ, ν, k) with fixed σ_1 and multiply this number by $\frac{1}{d!} \cdot |\{\sigma \in \mathcal{S}_d : \mathcal{C}(\sigma) = \mu\}|$ to obtain $\vec{H}_g^k(\mu, \nu)$. We can thus simplify this counting problem with a smart choice of σ_1 (see Equation (2)). We translate the counting problem to a problem of counting monodromy graphs as in [CJM11] and [DK16]. In the latter, the choice of σ_1 as in Equation (2) was already utilised.

To give our description of mixed Hurwitz numbers in terms of monodromy graphs, we make the following choice for fixed $\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$:

$$(2) \quad \sigma = (1 \cdots \mu_1)(\mu_1 + 1 \cdots \mu_1 + \mu_2) \cdots \left(\sum_{i=1}^{\ell(\mu)-1} \mu_i + 1 \cdots \sum_{i=1}^{\ell(\mu)} \mu_i \right),$$

where the cycle $\sigma_1^s = \left(\sum_{i=1}^{s-1} \mu_i + 1 \cdots \sum_{i=1}^s \mu_i \right)$ is labeled by s . We define $M_g^k(\mu, \nu)$ to be the number of mixed factorisations of type (g, μ, ν, k) with σ_1 as in Equation (2). The number of permutations of cycle type μ with labeled cycles is

$$\epsilon(\mu) = \frac{d!}{\mu_1 \cdots \mu_{\ell(\mu)}}$$

and we see that

$$(3) \quad H_g^k(\mu, \nu) = \frac{1}{d!} \epsilon(\mu) M_g^k(\mu, \nu) = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} M_g^k(\mu, \nu).$$

We will express $M_g^k(\mu, \nu)$ in terms of monodromy graphs. We begin by associating a graph to a mixed factorisation.

Construction 3.2. Let $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ be a mixed factorisation of type (g, μ, ν, k) with σ_1 as in Equation (2). We associate a graph with labeled vertices and edges and with a map to the interval $[0, \dots, m+1]$ as follows:

Constructing the graph.

- (1) We start with $\ell(\mu)$ vertices over 0, labeled by $\sigma_1^1, \dots, \sigma_1^{\ell(\mu)}$. We will call these vertices *in-ends*. Moreover, we attach an edge e_v to each vertex v over 0 which maps to $(0, 1)$. We label these edge attached to the vertex labeled σ_1^j by the same label.
- (2) We define $\Sigma_{i+1} = \tau_i \cdots \tau_1 \sigma_1$ for $i = 1, \dots, m$. We define $\Sigma_0 = \sigma_1$ and $\Sigma_{m+1} = (\sigma_2)^{-1}$. Comparing Σ_i and Σ_{i+1} , the transposition τ_i either joins two cycles of Σ_i or cuts one cycle in two.

Assuming the preimage of $[0, i)$ has been constructed, repeat the steps (3)–(4) until $i = m$:

- (3a) **[Join]** If τ_i joins the cycles Σ_{i-1}^s and $\Sigma_{i-1}^{s'}$ to a new cycle Σ' , we create a vertex over i labeled τ_i . This vertex is joined with the edges corresponding to Σ_{i-1}^s and $\Sigma_{i-1}^{s'}$. These edges map to some interval (a_s, i) and $(a_{s'}, i)$ respectively, where a_s (resp. $a_{s'}$) is the image of the other vertex adjacent to the edge corresponding to Σ_{i-1}^s (resp. $\Sigma_{i-1}^{s'}$). We call those edges the *incoming edges at τ_i* . Moreover, we attach an edge to τ_i mapping to $(i, i+1)$, which we label by Σ' . We call this edge the *outgoing edge at τ_i* .
- (3b) **[Cut]** If τ_i cuts Σ_{i-1}^s into Σ' and Σ'' , we create a vertex over i labeled τ_i . We attach one edge connecting τ_i to the edge corresponding to Σ_{i-1}^s , which maps to (a_s, i) as above and attach two edges mapping to $(i, i+1)$ labeled Σ' and Σ'' respectively. As above, we call the edge mapping to (a_s, i) the *ingoing edge at τ_i* and the edges mapping to $(i, i+1)$ *outgoing edges at τ_i* .
- (4) We extend those edges which so far are only adjacent to one vertex, such that the edge e maps to $(a_e, i+1)$, where a_e is the image of the vertex adjacent to e .

- (5) When $i = m$ is reached, the leaves of the graph which are not adjacent to in-ends correspond to the cycles of Σ_{m+1} . We create vertices over $m+1$ labeled $(\sigma_2^{-1})^1, \dots, (\sigma_2^{-1})^{\ell(\nu)}$ and connect the corresponding edges to those vertices.

Colouring the graph.

- (6) We color all edges *normal*.
 (7) We color all edges adjacent to in-ends *dashed*.
 (8) We repeat steps [(9a)] and [(9b)] for all transpositions τ_1, \dots, τ_k .
 (9a) **[Cut]** If τ_i is a transposition as in (3a), then we assume $\tau_i = (a \ b)$ with $a < b$ and $a \in \Sigma_{i-1}^s$ and $b \in \Sigma_{i-1}^{s'}$. Then we colour the edge labeled $\Sigma_{i-1}^{s'}$ *bold* and the outgoing edge at τ_i *dashed*.
 (9b) **[Join]** If τ_i is a transposition as in (3b), we colour the edge labeled Σ_{i-1}^s *bold*. For $\tau_i = (a \ b)$ with $a < b$, we colour the edge corresponding to the cycle containing b *dashed*.

Distributing counters.

We distribute a counter to all non-normal edges.

- (10) We start by distributing 1 to all edges adjacent to in-ends.
 (11) For $\tau_i = (r_i \ s_i)$, where $r_i < s_i$, there is a unique way of expressing s_i as follows:

$$s_i = \sum_{j=1}^l \mu_j + c,$$

where $c < \mu_{j+1}$. Then we distribute c to the outgoing dashed/bold edge adjacent to the vertex labeled τ_i .

Relabeling the graph.

- (12) We drop the labels τ_i at the vertices of $1, \dots, m$.
 (13) We label the in-ends (resp. out-ends) by $1, \dots, \ell(\mu)$ (resp. $1, \dots, \ell(\nu)$) according to the labels of σ_1 and σ_2 .
 (14) If a vertex or an edge is labeled by a cycle σ , we replace the label by the length of the cycle.

We obtain a graph Γ . We call Γ the *mixed monodromy graph of type (g, μ, ν, k) associated to $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$* .

Example 3.3. In Figure 1, in the upper picture we illustrate the cut-and-join process for the following mixed factorisation of type $(1, (2, 2), (4))$:

$$((12)(34), (12), (23), (13), (1243)).$$

In fact, it is a mixed factorisation of type $(1, (2, 2), (4), 3)$ and the associated mixed monodromy graph is illustrated in the lower picture.

For the remainder of this section, we will classify the graphs we obtain from Construction 3.2. Moreover, we will understand how many mixed factorisations yield the same monodromy graph. This result will be our main tool in the discussion of polynomiality.

Definition 3.4. A mixed monodromy graph Γ of type (g, μ, ν, k) is a graph with a map to $[0, m+1]$ (where $m = 2g - 2 + \ell(\mu) + \ell(\nu)$) with the following properties:

Graph/Map conditions.

- (1) The graph Γ is a connected.
 (2) The first Betti number of Γ is g .

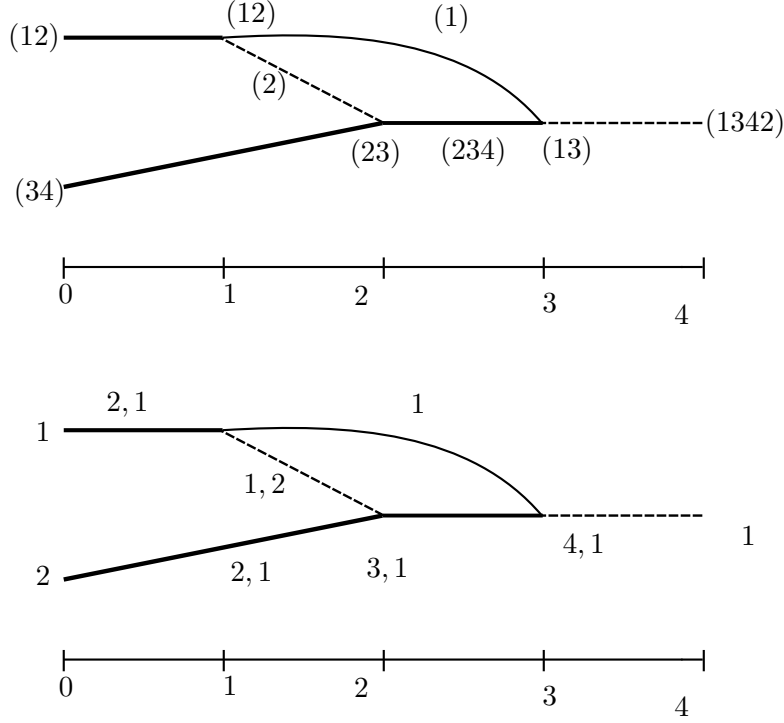


FIGURE 1. In the upper graph, the bold permutations correspond to transposition τ_i , the other ones correspond to the cycles of the permutations $\tau_i \cdots \tau_1 \sigma_1$. In the lower graph, the non-normal edges are bi-labeled, where the first number is the weight and the second is the counter.

- (3) The map sends vertices to integers, we call the image i of a vertex its position. Moreover, the map sends edges to open intervals. For a vertex of position i , we call edges mapped to (a, i) for $a < i$ *incoming edges at i* and edges mapped to (i, a) for $a > i$ *outgoing edges at i* .
- (4) The graph has $\ell(\mu) + \ell(\nu)$ leaves. There are $\ell(\mu)$ leaves mapped to 0 labeled by $1, \dots, \ell(\mu)$ and $\ell(\nu)$ leaves over $m + 1$ labeled by $1, \dots, \ell(\nu)$.
- (5) Over each integer $i \in [0, m]$, there is exactly one vertex which locally looks like one of the graph in Figure 2. We call these vertices *inner vertices*.

Weight conditions.

- (4) We assign a positive integer weight $\omega(e)$ to each edge e . The in-end labeled i has weight μ_i . The out-end labeled j has weight ν_j .
- (5) At each inner vertex, the sum of the weights of incoming edges equals the sum of the weights of outgoing edges.

This is known as the *balancing condition*. (The term balancing condition comes from the relation of these graphs to tropical geometry [CJM10]. Literature in non-archimedean geometry refers to maps satisfying a balancing condition as *harmonic morphisms* [Cap14] [ABBR15].)

Colouring conditions.

The following conditions are only applied to edges adjacent to the first k inner vertices.

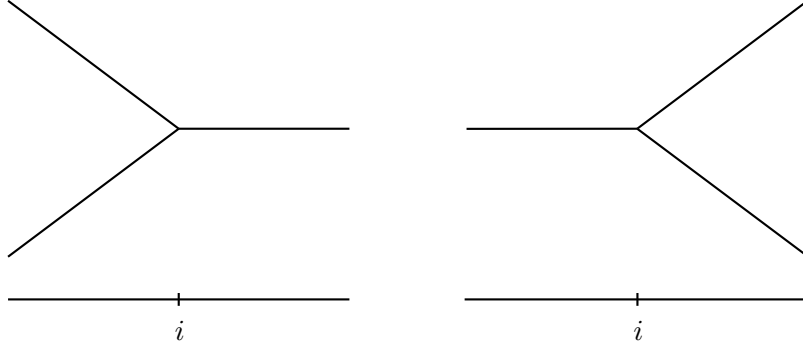


FIGURE 2. Local structure of the map for a monodromy graph.

- (6) We colour the edges of the graph by the three colours: normal, bold and dashed, such that each inner vertex is one of the six types in Figure 3.
- (7) There are no normal in-ends.
- (8) We call a connected path of bold edges beginning at an in-end a *chain*.
- (9) Let C and C' be two chains and let f_C (resp. $f_{C'}$) be the position of the first inner vertex of C (resp. C') and let l_C (resp. $l_{C'}$) be the position of the position of the last inner vertex of C (resp. C'). Then we require the intervals $[f_C, l_C]$ and $[f_{C'}, l_{C'}]$ to have empty intersection.
- (10) The intervals $[f_C, l_C]$ induce a natural ordering on the chains, namely $C < C'$ if $f_C < f_{C'}$. We require this ordering to be compatible with the ordering of the partition μ as follows: Let C_1 and C_2 be two chains of bold edges, i_1 and i_2 the respective in-ends, then we demand $C_1 < C_2$ if and only if $i_1 < i_2$.

The ordering of the chains corresponds to the monotonicity condition as we will see later.

Counter conditions.

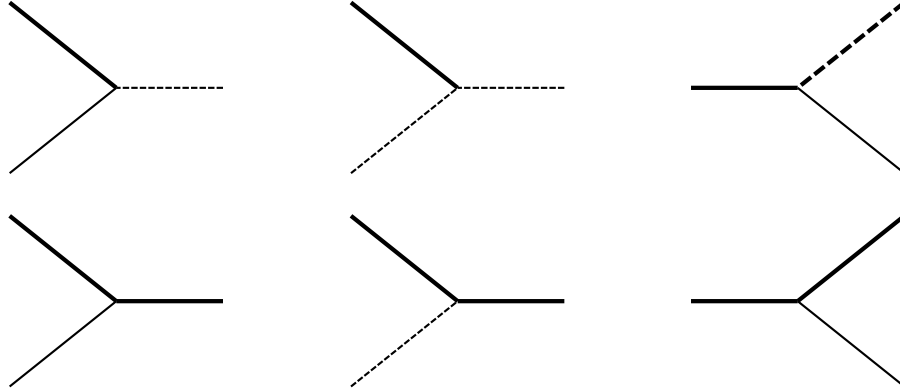


FIGURE 3. Local colouring of the graph.

- (11) We distribute a counter to each non-normal edge (thus, those inner edges are bi-labeled by the weight and the counter and the non-normal leaves are tri-labeled where the additional label is a number in $\{1, \dots, \ell(\mu)\}$ or $\{1, \dots, \ell(\nu)\}$).
- (12) The counter for each in-end is set to 1.

- (13) At each inner vertex v , there is a unique incoming bold edge i_v and a unique out-going non-normal edge o_v . We require the counter of o_v to be bigger equal than the counter of i_v .
- (14) Every non-normal edge arises from a unique chain of bold edges: Every bold edge is part of a unique chain and every dashed edge is sourced at a unique chain. Let the non-normal edge e arise from the chain starting at the in-end labeled i . The counter l_e of the non-normal edge e is smaller or equal than μ_i and greater than $\mu_i - \omega(e)$.

The last condition reflects that these cycles corresponding to each such edge e should contain at least $\mu_i - (l_e - 1)$ elements.

Definition 3.5. Let Γ be a mixed monodromy graph of type (g, μ, ν, k) . We call the graph we obtain by removing the counters the *reduced monodromy graph* of Γ .

A graph Γ that appears as a mixed monodromy graph of type (g, μ, ν, k) without counters is called a *reduced monodromy graph of type (g, μ, ν, k)* .

We called the graph we obtained from Construction 3.2 a mixed monodromy graph. The following lemma justifies the choice of this term.

Lemma 3.6. *The graphs obtained from Construction 3.2 are mixed monodromy graphs in the sense of Definition 3.4.*

Proof. Let $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ be a mixed transposition of type (g, μ, ν, k) with σ_1 as in Equation (2). The conditions (1)-(7) in Definition 3.4 follow immediately by construction.

The chains of bold edges correspond to the following situation in the symmetric group setting: Suppose $\tau_i = (r_i \ s_i)$ for $r_i < s_i$. Since we chose σ_1 in Equation (2), we can group the transpositions τ_i for $i \leq k$. We say τ_i is of type t , if s_i is contained in the cycle of σ_1 labeled t . Now, for $i < j$, let t_i (resp. t_j) be the type of τ_i (resp. τ_j), then $t_i \leq t_j$.

A chain of bold edges starting at the in-end i corresponds to the transpositions of type i . Thus conditions (8)-(10) follow.

The counter conditions (11)-(13) follow by construction. For condition (14) we observe the following: Let e be a non-normal edge which arises from the chain of bold edges C and whose source vertex has position p . Let C start at i and let l_e be the counter of e . Moreover, let the i -th cycle of σ_1 is of the form

$$\left(\sum_{a=1}^{i-1} \mu_a + 1 \cdots \sum_{a=1}^i \mu_a \right).$$

Then for $\tau_p = (r_p \ s_p)$ we have $s_p = \sum_{a=1}^{i-1} \mu_a + l_e$. By monotonicity multiplying σ_1 by $\tau_1 \cdots \tau_p$ does not change the images of $\sum_{a=1}^{i-1} \mu_a + l_e, \dots, \sum_{a=1}^i \mu_a - 1$. Thus the cycle of $\tau_p \cdots \tau_1 \sigma_1$ containing $\sum_{a=1}^{i-1} \mu_a + l_e$ has the following structure

$$\left(\cdots \sum_{a=1}^{i-1} \mu_a + l_e \cdots \sum_{a=1}^i \mu_a \cdots \right),$$

where the dots left and right indicate other elements. Thus the weight $\omega(e)$ of the weight e fulfils the following inequality

$$\omega(e) \geq \mu_i - l_e + 1$$

or equivalently

$$l_e > \mu_i - \omega(e).$$

Thus condition (14) is fulfilled as well. \square

Definition 3.7. An automorphism of a mixed monodromy graph Γ is a graph automorphism $f : \Gamma \rightarrow \Gamma$, such that:

- (1) The function f respects weights, colours and counters.
- (2) The following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Gamma \\ \downarrow & & \downarrow \\ [0, m+1] & \xrightarrow{\text{id}} & [0, m+1]. \end{array}$$

We denote the automorphism group of Γ by $\text{Aut}(\Gamma)$.

We are now ready to give a weighted bijection between mixed factorisations and mixed monodromy graphs of type (g, μ, ν, k) .

Lemma 3.8. *Let Γ be a mixed monodromy graph of type (g, μ, ν, k) . The number $m(\Gamma)$ of mixed factorizations of type (g, μ, ν, k) with σ_1 as in Equation (2) for which Construction 3.2 produces Γ is*

$$m(\Gamma) = \frac{1}{|\text{Aut}(\Gamma)|} \prod \omega(e),$$

where we take the product over all dashed and normal edges e , which are not adjacent to out-ends.

We call $m(\Gamma)$ the multiplicity of Γ .

Remark 3.9. An immediate consequence of this lemma is that the number $m(\Gamma)$ does not depend on the counters of Γ . We will use this in Section 4.

Proof. Let v be one of the first k inner vertices. If v is a cut, the corresponding transposition is uniquely defined by the weights of the outgoing edges and the counter of the outgoing dashed or bold edge. If two edges are joined at v , the larger entry of the corresponding transposition is uniquely defined by the counter of the outgoing non-normal edge and the source chain of the in-going bold edge. However, we have a number of possibilities for the first element of the transposition, which is exactly the weight of the non-bold ingoing edge.

Now let v be an inner vertex whose position is greater than k . If v is a cut with ingoing edge e , there are $\omega(e)$ possibilities for τ_v , except when $\omega(e) = 2n$ and both outgoing edges have weight n . Then, there are only n possibilities for τ_v . If both cycles have distinguishable evolution, it matters which cycle has which evolution and obtain a factor of 2. If the cycles have undistinguishable evolution, this corresponds to a contribution of $\text{Aut}(\Gamma)$.

If v is a join with ingoing edges e and e' , the number of possibilities for τ_v is $\omega(e) \cdot \omega(e')$. Thus the lemma is proved. \square

Example 3.10. The multiplicity of the graph in the right of Figure 1 is 2.

By our previous discussion we can compute mixed Hurwitz numbers in terms of mixed monodromy graphs.

Proposition 3.11. *Let g be a non-negative integer, k a positive integer and μ, ν ordered partitions of the same number d . Then:*

$$H_g^k(\mu, \nu) = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} \sum_{\Gamma} m(\Gamma),$$

where we sum over all mixed monodromy graphs of type (g, μ, ν, k) .

Proof. This is an immediate consequence of Lemma 3.6, Lemma 3.8 and Equation (3). \square

4. PIECEWISE POLYNOMIALITY

We want to use Lemma 3.8 to study the piecewise polynomiality of mixed Hurwitz numbers in the flavour of the discussion of Section 6 of [CJM10]. We begin by studying the genus 0 case and we will use Erhart theory to generalize these results to higher genera.

4.1. The genus 0 case. The aim of this subsection is to show that M_0^k is piecewise polynomial and to provide a constructive method to compute the polynomials in each chamber. By Equation (3) this also produces a method to compute the polynomials for H_0^k in each chamber.

Proposition 4.1. *The function*

$$M_0^k : \mathbb{N}^{\ell(\mu)} \times \mathbb{N}^{\ell(\nu)} \rightarrow \mathbb{Q}$$

$$(\mu, \nu) \mapsto M_0^k(\mu, \nu)$$

is piecewise polynomial, i.e. for every chamber C induced by the hyperplane arrangement \mathcal{W} , there exists a polynomial $m_0^k(C) \in \mathbb{Q}[\underline{M}, \underline{N}]$, such that $M_0^k(\mu, \nu) = m_0^k(C)(\mu, \nu)$ for all $(\mu, \nu) \in C$.

Proof. The proof follows from Lemma 4.5, Corollary 4.6 and Lemma 4.8. \square

Remark 4.2. Note, that this Proposition does not prove that the functions $h_0^k(C)$ are polynomials. It follows from Theorem 2.6, that the polynomials $m_0^k(C)$ contain a factor of $\prod_{i=1}^{\ell(\mu)} \mu_i$ — however this is not true for the contribution of each graph to $m_0^k(C)$ as can be seen in Example 4.10. It would be interesting to see how the contributions for each graph add up to a polynomial which is divisible by $\prod \mu_i$.

We start by examining the edge weights in the equation

$$(4) \quad M_0^k(\mu, \nu) = \sum_{\Gamma} m(\Gamma) = \sum_{\Gamma} \prod \omega(e),$$

where we sum over all mixed monodromy graphs of type $(0, \mu, \nu, k)$. We want to group some of the graphs appearing in the sum.

Definition 4.3. Let Γ be a reduced monodromy graph of type (g, μ, ν, k) . Then we define the function

$$F(\Gamma, \mu, \nu) = \left| \{ \text{Mixed monodromy graphs } \tilde{\Gamma} \text{ with reduced monodromy graph } \Gamma \} \right|.$$

Then we can rewrite Equation (4) as follows

$$(5) \quad M_0^k(\mu, \nu) = \sum_{\Gamma} \prod \omega(e) F(\Gamma, \nu, \nu),$$

where we sum over all reduced monodromy graphs of type $(0, \mu, \nu, k)$.

For the remainder of this subsection, we prove the following three claims constructively:

- (1) The set of reduced monodromy graphs of type $(0, \mu, \nu, k)$ only depends on the chamber C (induced by the hyperplane arrangement \mathcal{W} in which μ and ν are contained and not on the specific entries μ_i, ν_j . (Lemma 4.5)
- (2) The product $\prod \omega(e)$ appearing in Equation (5) is a polynomial. (Corollary 4.6)
- (3) The function $F(\Gamma, \nu, \nu)$ is a polynomial in each chamber. (Lemma 4.8)

These steps will yield a constructive method of computing the polynomials $m_0^k(C)$. Lemma 4.8 represents an ingredient that was not used in the literature in this context before. Claim (1) was actually observed in [CJM10] for the case of (non-mixed) monodromy graphs. For the convenience of the reader, we repeat the argument. We begin by introducing some notation.

Notation 4.4. Let μ be a partition and let $I \subset \{1, \dots, \ell(\mu)\}$. Then μ_I is the subpartition of μ given by $\mu_I = (\mu_{i_1}, \dots, \mu_{i_{|I|}})$, where $i_j < i_{j+1}$.

Lemma 4.5. *The set of reduced monodromy graphs of type $(0, \mu, \nu, k)$ only depends on the chamber C .*

Proof. Let Γ be a mixed monodromy graph of type $(0, \mu, \nu, k)$, then we cut Γ along e and obtain two mixed monodromy graphs Γ_1 and Γ_2 . Let e point away from Γ_1 , then Γ_1 and Γ_2 are of respective type $(0, \mu_{I_1}, \nu_{J_1} \cup \{\omega(e)\}, k_1)$ and $(0, \mu_{I_2} \cup \{\omega(e)\}, \nu_{J_2}, k_2)$ for subsets $I_1, I_2 \subset \{1, \dots, \ell(\mu)\}$ and $J_1, J_2 \subset \{1, \dots, \ell(\nu)\}$. and $k_1 + k_2 = k$. Moreover, we have $|\mu_{I_1}| = |\nu_{J_1} \cup \{\omega(e)\}|$ and we obtain

$$\omega(e) = \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j.$$

The only requirement for a reduced monodromy graph to contribute to the sum Equation (5) is the positivity of all edge weights. As we saw above, this only depends on the chamber C we pick. \square

Claim (2) immediately follows:

Corollary 4.6. *Every edge weight $\omega(e)$ is a linear polynomial in the entries of μ and ν . Thus $\prod \omega(e)$ is a polynomial in the entries of μ and ν as well.* \square

Before we can prove claim (3), we need the following Definition.

Definition 4.7. Let B a path in Γ starting at an in-end, such that

- (1) There are s edges in B .
- (2) The first $s - 1$ edges form a chain of bold edges. (see (8) of Definition 3.4)
- (3) The last edge is dashed.

We call B a *chain-path of length s* .

The following lemma is our key step towards Proposition 4.1.

Lemma 4.8. *The function $F(\Gamma, \mu, \nu)$ can be expressed as a polynomial in each chamber C*

Proof. We fix a reduced monodromy graph Γ of type $(0, \mu, \nu, k)$. Assigning counters to Γ translates to assigning counters to each chain-path in Γ as follows: Fix a chain-path B of length s and distribute the counter $l_{\tilde{k}}$ to the \tilde{k} -th edge $e_{\tilde{k}}$ in B . Moreover, let B start at the in-end labeled i . Then (l_1, \dots, l_s) satisfies the counter conditions if and only if

- (1) $l_1 \leq \dots \leq l_s$ (see condition (13) in Definition 3.4)
- (2) $1 \leq l_1 \leq \mu_i$ (see condition (14) in Definition 3.4)
- (3) $\max\{1, \mu_i - \omega(e_{\tilde{k}})\} \leq l_{\tilde{k}} \leq \mu_i$. (see condition (14) in Definition 3.4)

Thus we need to prove, that the cardinality of the set

$$(6) \quad \{(l_1, \dots, l_s) | l_1 \leq \dots \leq l_s, l_1 = 1, \max\{1, \mu_i - \omega(e_{\tilde{k}})\} \leq l_{\tilde{k}} \leq \mu_i\}$$

is piecewise polynomial in the entries of μ and ν . We can express this cardinality as the following iterative sum

$$\sum_{\substack{\mu_i \\ l_2 = \\ \max\{1, \mu_i - \omega(e_1)\}}} \sum_{\substack{\mu_i \\ l_3 = \\ \max\{l_2, \mu_i - \omega(e_2)\}}} \cdots \sum_{\substack{\mu_i \\ l_{s-1} = \\ \max\{l_{s-2}, \mu_i - \omega(e_{s-1})\}}} \mu_i - \max\{l_{s-1}, \mu_i - \omega(e_s)\}.$$

If we know whether $\max\{\mu_i - \omega(e_1), 1\} = \mu_i - \omega(e_1)$ and if we have a total ordering on the $\mu_i - \omega(e_{\tilde{k}})$, we can compute this sum using Faulhaber's formula

$$\sum_{\tilde{k}=1}^n \tilde{k}^p = \frac{1}{p+1} \sum_{j=0}^p (-1)^j \binom{p+1}{j} B_j n^{p+1-j},$$

where B_j is the j -th Bernoulli number. Notice, that the right-hand side is a polynomial in n . Thus, the cardinality of the set in Equation (6) is a polynomial in μ_i and the edge weights $\omega(e)$ (and since $\omega(e)$ is linear form in the entries of μ and ν , the cardinality is a polynomial in the entries of μ and ν), whenever we know the value of $\max\{\mu_1 - \omega(e_1), 1\}$ and if we have a total ordering on the $\mu_i - \omega(e_{\tilde{k}})$, which we can compute iteratively. Now, we show that choosing a chamber C for μ and ν implies those conditions.

Let $I_{\tilde{k}} \subset \{1, \dots, \ell(\mu)\}$ and $J_{\tilde{k}} \subset \{1, \dots, \ell(\nu)\}$ for $\tilde{k} = 1, \dots, s$, such that

$$\omega(e_{\tilde{k}}) = \sum_{j \in I_{\tilde{k}}} \mu_j - \sum_{j \in J_{\tilde{k}}} \nu_j$$

for all \tilde{k} . We observe that for the edge e_1 , we get

$$\max\{1, \mu_i - \omega(e_1)\} = \mu_i - \omega(e_1)$$

if and only if

$$\sum_{j \in J_1} \nu_j - \sum_{j \in I_1 - \{i\}} \mu_j > 0.$$

This implies that in a fixed chamber C , we know the value of $\max\{1, \mu_i - \omega(e_1)\}$. Moreover, we fix two edges e_j and $e_{\tilde{k}}$, such that $j < \tilde{k}$. We see that since e_j and $e_{\tilde{k}}$ are in the same chain-path and $e_{\tilde{k}}$ appears later than e_j , we have $I_j \subset I_{\tilde{k}}$ and $J_j \subset J_{\tilde{k}}$. Thus $\omega(e_{\tilde{k}}) > \omega(e_j)$ if and only if

$$\sum_{l \in I_{\tilde{k}} - I_j} \mu_l - \sum_{l \in J_{\tilde{k}} - J_j} \nu_l > 0.$$

Thus we can answer whether $\omega(e_{\tilde{k}}) > \omega(e_j)$ in each chamber.

Let $P_B(C)$ be the polynomial computing the cardinality of the set in Equation (6) associated to the chain-path B in the chamber C . Since we can choose counters in each chain-path independently (they do not intersect, since chains of bold edges do not intersect), in the chamber C , the function $F(\Gamma, \mu, \nu)$ is given by $\prod P_B(C)$, where we take the product over all chain-paths. Since the graph is finite, $F(\Gamma, \mu, \nu)$ is a polynomial in the entries of μ and ν in each chamber C as desired. \square

We have now proved the following Algorithm, computing the polynomials $h_0^k(C)$.

Algorithm 4.9.

Input : Two positive integer $\ell(\mu)$, $\ell(\nu)$, a non-negative integer k and a chamber C induced by \mathcal{W} .

Output: The polynomial $h_0^k(C)$.

Compute the set $\mathfrak{G}(\ell(\mu), \ell(\nu), C, k)$ of all monodromy graphs of type $(0, \mu, \nu, k)$ in C ;

for $\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, k)$ **do**

 Compute the set $E(\Gamma)$ of all edges;

for $e \in E(\Gamma)$ **do**

 Express the weight $\omega(e)$ of e as a linear form in the entries of μ and ν ;

end

 Compute the polynomial $W(\Gamma) = \prod \omega(e)$, where the product is taken over all non-bold edges which are not adjacent to out-ends;

 Compute the set $CP(\Gamma)$ of all chain-paths in Γ ;

for $P \in CP(\Gamma)$ **do**

 Compute the polynomial q_P expressing

$$\sum_{l_2 = \max\{1, \mu_i - \omega(e_1)\}}^{\mu_i} \sum_{l_3 = \max\{l_2, \mu_i - \omega(e_2)\}}^{\mu_i} \cdots \sum_{l_{s-1} = \max\{l_{s-2}, \mu_i - \omega(e_{s-1})\}}^{\mu_i} \mu_i - \max\{l_{s-1}, \mu_i - \omega(e_s)\} \text{ in } C;$$

end

 Compute the polynomial $C(\Gamma) = \prod_{P \in CP(\Gamma)} q_P$;

 Compute the polynomial $m(\Gamma) = W(\Gamma) \cdot C(\Gamma)$;

end

Compute the polynomial $h_0^k(C) = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} \sum_{\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, k)} m(\Gamma)$;

return $h_0^k(C)$;

□

Example 4.10. We use this algorithm to compute the polynomials for h_0^2 for $\ell(\mu) = \ell(\nu) = 2$. The possible graphs are illustrated in Figure 4. There are four chambers in that case as illustrated in Figure 9 in [CJM10].

We start with the chamber C_1 given by $\mu_1 > \nu_1, \mu_1 > \nu_2, \mu_2 < \nu_1, \mu_2 < \nu_2$. In this chamber the graphs I.a, I.b, IV, V, VI, VII contribute positive multiplicities:

$$\begin{aligned} \text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\ \text{mult(I.b)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\ \text{mult(IV)} &= (\mu_1 - \nu_2) \mu_2 (\mu_1 - \nu_2), \\ \text{mult(V)} &= (\mu_1 - \nu_1) \mu_2 (\mu_1 - \nu_1), \\ \text{mult(VI)} &= (\mu_1 - \nu_2) \mu_2 \nu_2, \\ \text{mult(VII)} &= (\mu_1 - \nu_1) \mu_2 \nu_1. \end{aligned}$$

Adding all these contributions we obtain

$$m_0^2(C_1) = \mu_1 \mu_2 (2\mu_1 + \mu_2 - \nu_1 - \nu_2 + 1).$$

Next we look at the chamber C_2 given by $\mu_1 < \nu_1, \mu_1 > \nu_2, \mu_2 < \nu_1, \mu_2 > \nu_2$. In this chamber the graphs I.a, I.b, IV, V contribute positive multiplicities:

$$\begin{aligned} \text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\ \text{mult(I.b)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \frac{\nu_2^2}{2} + \frac{\nu_2}{2} \right), \\ \text{mult(III)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \mu_2 \nu_2 + \frac{\nu_2^2}{2} - \frac{\nu_2}{2} \right), \\ \text{mult(IV)} &= (\mu_1 - \nu_2) \mu_2 \nu_2, \\ \text{mult(VI)} &= (\mu_1 - \nu_2) \mu_2 (\mu_1 - \nu_2). \end{aligned}$$

Adding all these contributions we obtain

$$m_0^2(C_2) = \mu_1 \mu_2 \left(\frac{3}{2} \mu_2 + \frac{3}{2} - 2\nu_2 + \mu_1 \right).$$

Let the chamber C_3 be given by $\mu_1 < \nu_1, \mu_1 < \nu_2, \mu_2 > \nu_1, \mu_2 > \nu_2$. In this chamber the graphs I.a, I.b, II, III contribute positive multiplicities:

$$\begin{aligned} \text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \frac{\nu_1^2}{2} + \frac{\nu_1}{2} \right), \\ \text{mult(I.b)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \frac{\nu_2^2}{2} + \frac{\nu_2}{2} \right), \\ \text{mult(II)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \mu_2 \nu_1 + \frac{\nu_1^2}{2} - \frac{\nu_1}{2} \right), \\ \text{mult(III)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \mu_2 \nu_2 + \frac{\nu_2^2}{2} - \frac{\nu_2}{2} \right). \end{aligned}$$

Adding all these contributions we obtain

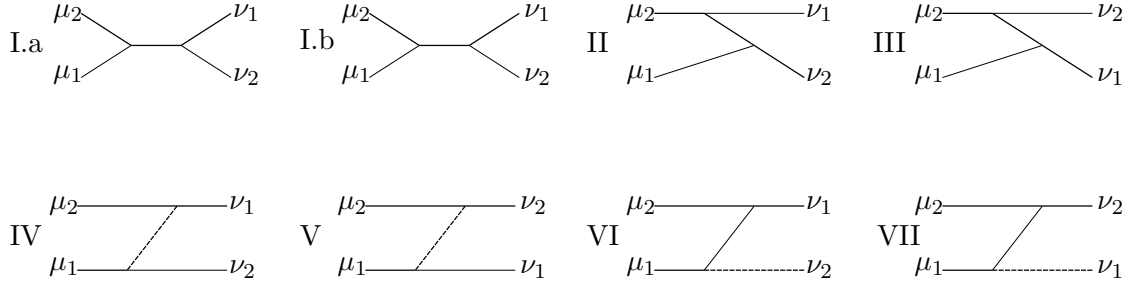
$$m_0^2(C_3) = \mu_1 \mu_2 (2\mu_2 + 2 - \nu_1 - \nu_2).$$

Lastly, we consider the chamber C_4 given by $\mu_1 > \nu_1, \mu_1 < \nu_2, \mu_2 > \nu_1, \mu_2 < \nu_2$. In this chamber the graphs I.a, I.b, II, V contribute positive multiplicities:

$$\begin{aligned} \text{mult(I.a)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \frac{\nu_1^2}{2} + \frac{\nu_1}{2} \right), \\ \text{mult(I.b)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} \right), \\ \text{mult(II)} &= \mu_1 \left(\frac{\mu_2^2}{2} + \frac{\mu_2}{2} - \mu_2 \nu_1 + \frac{\nu_1^2}{2} - \frac{\nu_1}{2} \right), \\ \text{mult(V)} &= (\mu_1 - \nu_1) \mu_2 \nu_2, \\ \text{mult(VII)} &= (\mu_1 - \nu_1) \mu_2 (\mu_1 - \nu_1). \end{aligned}$$

Adding all these contributions we obtain

$$m_0^2(C_4) = \mu_1 \mu_2 \left(\frac{3}{2} \mu_2 + \frac{3}{2} - 2\nu_1 + \mu_1 \right).$$

FIGURE 4. The graphs appearing for $\mathcal{H}_0((\mu_1, \mu_2), (\nu_1, \nu_2))$.

Thus we see

$$\begin{aligned} h_0^2(C_1) &= 2\mu_1 + \mu_2 - \nu_1 - \nu_2 + 1, & h_0^2(C_2) &= \frac{3}{2}\mu_2 + \frac{3}{2} - 2\nu_2 + \mu_1 \\ h_0^2(C_3) &= 2\mu_2 + 2 - \nu_1 - \nu_2, & h_0^2(C_4) &= \frac{3}{2}\mu_2 + \frac{3}{2} - 2\nu_1 + \mu_1 \end{aligned}$$

4.2. Piecewise polynomiality in arbitrary genus. For higher genera, we once again study

$$\mathcal{M}_g^k(\mu, \nu) = \sum m(\Gamma).$$

Let Γ be a mixed monodromy graph of type (g, μ, ν, k) . We introduce a variable x_i for each of the g cycles in Γ . Then by a similar argument as before, each edge weight may be expressed as a linear polynomial in the entries of μ and ν and in the x_i . We still require all edge weights $\omega(e)$ to be greater than zero and thus obtain a hyperplane arrangement H in the entries of μ and ν and x_i . We will refine this hyperplane arrangement along the way and obtain the piecewise polynomiality result that way.

As in the genus 0 case, we rewrite the equation above by passing over to reduced monodromy graphs:

$$(7) \quad M_g^k(\mu, \nu) = \sum_{\Gamma} \prod \omega(e) F(\Gamma, \nu, \nu),$$

where we sum over all reduced monodromy graphs of type (g, μ, ν, k) . Since there are additional variables x_i the order on each chain-path is no longer determined just by the entries of μ and ν . Thus, we need to refine the sum further. We begin by restricting to chambers due to a generalisation of Lemma 4.5 which was proved in Theorem 3.9 in [CJM11]:

Theorem ([CJM11]). *The set of reduced monodromy graphs of type (g, μ, ν, k) only depends on the chamber C .*

Thus, we know that $\prod \omega(e)$ is a polynomial in the x_i and the entries of μ and ν in each chamber. Now, we want to express $F(\Gamma, \nu, \nu)$ as a polynomial as well. The key point in the proof of Lemma 4.8 was the fact that the graph structure imposed an ordering on the edge weights $\omega(e)$ and 1. For higher genera this is no longer true as depending on the values of x_i there may be several orderings on each chain path (see Example 4.18). To deal with this problem, we introduce the notion of an ordering on a reduced monodromy graph.

Definition 4.11. An ordering O on a reduced monodromy graph Γ is a partial ordering on the edge weights and 1, that restricted to each chain path and 1 is a total ordering. We denote by $\mathcal{O}(\Gamma)$ the possible orderings on Γ .

Next, we refine the function $F(\Gamma, \mu, \nu)$.

Definition 4.12. Let Γ be a reduced monodromy graph and O an ordering on Γ . Then we define $F(\Gamma, \mu, \nu, \underline{x}, O)$ (where $\underline{x} = x_1, \dots, x_g$) to be the function counting all possible counter distributions on Γ compatible with O .

We want to argue that $F(\Gamma, \mu, \nu, \underline{x}, O)$ is a polynomial in the x_i and the entries of μ and ν . However, we have to be careful about the values of \underline{x} , since not all choices of \underline{x} are compatible with O . Thus, we define $Q(\Gamma, \mu, \nu, O)$ to be the set of all values for x_i fulfilling the ordering O . It is easy to see, that this set is convex and the x_i are bounded since all edge weights have to be positive. We thus obtain the following lemma:

Lemma 4.13. *The set $Q(\Gamma, \mu, \nu, O)$ is a polytope with equations given by linear forms in the entries of μ and ν .*

Definition 4.14. We denote the hyperplane arrangement in $\mathbb{N}^{\ell(\mu)+\ell(\nu)}$ induced by the combinatorial types of $Q(\Gamma, \mu, \nu, O)$ by $\mathcal{V}(\Gamma, O)$

By same argument as in Lemma 4.8, we also get the following lemma:

Lemma 4.15. *The function $F(\Gamma, \mu, \nu, \underline{x}, O)$ is a polynomial in \underline{x} and the entries of μ and ν for $\underline{x} \in Q(\Gamma, \mu, \nu, O)$.*

We can now rewrite Equation (7) as follows:

$$(8) \quad M_g^k(\mu, \nu) = \sum_{\Gamma} \sum_{O \in \mathcal{O}(\Gamma)} \sum_{\underline{x} \in Q(\Gamma, \mu, \nu, O)} \prod \omega(e) F(\Gamma, \mu, \nu, \underline{x}, O).$$

It is well-known that summing a polynomial over a polytope with rational vertices yields a quasi-polynomial (see e.g. [Woo14], [BBDL⁺14]).

Since $\prod \omega(e) F(\Gamma, \mu, \nu, \underline{x}, O)$ is a polynomial in \underline{x} and the entries of μ and ν and since $Q(\Gamma, \mu, \nu, O)$ is a polytope, $M_g^k(\mu, \nu)$ is a quasi-polynomial in each chamber of the hyperplane arrangement given as the common refinement of \mathcal{W} and the family $(\mathcal{V}(\Gamma, O))_{\Gamma, O}$.

Remark 4.16. We note that with our method, we only proved that we obtain a quasi-polynomial for $h_g^k(C)$. However, we know by Theorem 2.6 that the mixed Hurwitz number is a polynomial. Moreover, in Algorithm 4.17 we pick one chamber C' induced by the refined hyperplane arrangement in C . However, by Theorem 2.6 the result does not depend on the choice of the finer chamber in C .

We can now state our Algorithm for higher genera, computing the polynomials $h_g^k(C)$.

Algorithm 4.17.

Data: Two positive integer $\ell(\mu)$, $\ell(\nu)$, two non-negative integer k, g and a chamber C induced by \mathcal{W} .

Result: The polynomial $h_g^k(C)$.

Compute the set $\mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)$ of all monodromy graphs of type (g, μ, ν, k) in C ;
for $\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)$ **do**
 Compute the set $\mathcal{O}(\Gamma)$ of all orderings on Γ ;
 for $O \in \mathcal{O}(\Gamma)$ **do**
 Compute the polytope $Q(\Gamma, \mu, \nu, O)$ induced by the inequalities for Γ in C and O ;
 Compute the hyperplane arrangement $\mathcal{V}(\Gamma, O)$ induced by the equations for $Q(\Gamma, \mu, \nu, O)$;
 end
 Compute the common refinement $C(\mathcal{O})$ of C and the family $\mathcal{V}(\Gamma, O)_{O \in \mathcal{O}(\Gamma)}$;
end
Compute the common refinement $C(\mu, \nu)$ of C and the family $(C(\mathcal{O}(\Gamma)))_{\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)}$;
Choose some chamber C' in $C(\mathcal{O}(\Gamma))$;
for $\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)$ **do**
 Compute the set $E(\Gamma)$ of all edges in Γ ;
 for $e \in E(\Gamma)$ **do**
 Compute the weight $\omega(e)$ as a linear form in the entries of μ , ν and \underline{x} ;
 end
Compute the polynomial $W(\Gamma) = \prod \omega(e)$, where we take the product over all non-bold edges $e \in E(\Gamma)$ which are not adjacent to out-ends;
Compute the set $CP(\Gamma)$ of all chain-paths;
 for $O \in \mathcal{O}(\Gamma)$ **do**
 for $P \in CP(\Gamma)$ **do**
 Compute the polynomial $q_P(O)$ expressing

$$\sum_{\substack{\mu_i \\ l_2 = \max\{1, \mu_i - \omega(e_1)\}}} \sum_{\substack{\mu_i \\ l_3 = \max\{l_2, \mu_i - \omega(e_2)\}}} \cdots \sum_{\substack{\mu_i \\ l_{s-1} = \max\{l_{s-2}, \mu_i - \omega(e_{s-1})\}}} \mu_i - \max\{l_{s-1}, \mu_i - \omega(e_s)\}$$
 respecting the order O ;
 end
 Compute $c(O) = \prod_{P \in CP(\Gamma)} q_P(O)$;
 end
 Compute $m(\Gamma, C') = \sum_{O \in \mathcal{O}(\Gamma)} \sum_{\underline{x} \in Q(\Gamma, \mu, \nu, O)} \prod W(\Gamma) c(O)$;
end
Compute $h_g^k(C) = \frac{1}{\mu_1 \cdots \mu_{\ell(\mu)}} \sum_{\Gamma \in \mathfrak{G}(\ell(\mu), \ell(\nu), C, g, k)} m(\Gamma, C')$;
return $h_g^k(C')$;

□

Example 4.18. In this example, we treat the graph Γ in Figure 5. The weight function is

$$\mu_1 \mu_2 (\mu_2 + \mu_3 - x_2).$$

The only chain path is given by $(\mu_3, \mu_2 + \mu_3, x_1, \mu_1 + x_1, \nu_1)$, thus the counter function is given by:

$$F(\Gamma, \mu, \nu, \underline{x}) = \sum_{l_2=1}^{\mu_3} \sum_{\substack{l_3=\max \\ \{l_2, \mu_3-x_1+1\}}}^{\mu_3} \sum_{\substack{l_4=\max \\ \{l_3, \mu_3-\mu_1-x_1+1\}}}^{\mu_3} \mu_3 - l_4 + 1.$$

There are five different ordering:

$$\begin{aligned} O_1 : \nu_1 &> \mu_2 + \mu_3 > \mu_1 + x_1 > x_1 > \mu_3 \\ O_2 : \nu_1 &> \mu_2 + \mu_3 > \mu_1 + x_1 > \mu_3 > x_1 \\ O_3 : \nu_1 &> \mu_2 + \mu_3 > \mu_3 > \mu_1 + x_1 > x_1 \\ O_4 : \nu_1 &> \mu_1 + x_1 > \mu_2 + \mu_3 > x_1 > \mu_3 \\ O_5 : \nu_1 &> \mu_1 + x_1 > \mu_2 + \mu_3 > \mu_3 > x_1 \end{aligned}$$

We show, how to compute the contributions for O_1 and O_2 . For O_1 we obtain:

$$F(\Gamma, \mu, \nu, \underline{x}, O_1) = \sum_{l_2=1}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1.$$

For the ordering O_2 , we get the following formula:

$$F(\Gamma, \mu, \nu, \underline{x}, O_2) = \sum_{l_2=1}^{\mu_3-x_1+1} \sum_{l_3=\mu_3-x_1}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1 + \sum_{l_2=\mu_3-x_1+2}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1.$$

The ordering impose the following inequalities on x_1 :

$$\begin{aligned} O_1 : \mu_2 + \mu_3 - \mu_1 &> x_1 > \mu_3 \\ O_2 : \min\{\mu_2 + \mu_3 - \mu_1, \mu_1\} &> x_1 > \max\{0, \mu_3 - \mu_1\} \end{aligned}$$

The inequality given by O_2 induces additional hyperplanes not given by equations of type $\sum : i \in I\mu_i - \sum_{j \in J} \nu_j$. The contributions of O_1 and O_2 (which we do not expand further, since the first sum alone expands to 20 terms) are

$$\begin{aligned} &\sum_{x_1=\mu_3}^{\mu_2+\mu_3-\mu_1} \left(\mu_1 \mu_2 (\mu_2 + \mu_3 - x_2) \sum_{l_2=1}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1 \right) + \\ &\sum_{x_1=\max\{0, \mu_3-\mu_1\}}^{\min\{\mu_2+\mu_3-\mu_1, \mu_1\}} \left(\mu_1 \mu_2 (\mu_2 + \mu_3 - x_2) \sum_{l_2=1}^{\mu_3-x_1+1} \sum_{l_3=\mu_3-x_1}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1 + \right. \\ &\quad \left. \sum_{l_2=\mu_3-x_1+2}^{\mu_3} \sum_{l_3=l_2}^{\mu_3} \sum_{l_4=l_3}^{\mu_3} \mu_3 - l_4 + 1 \right), \end{aligned}$$

which is a polynomial in each chamber of the refined hyperplane arrangement. We note that after computing the polynomial for every graph our method yields the same polynomial in each chamber (see Remark 4.16), while this may not be true of each graph.

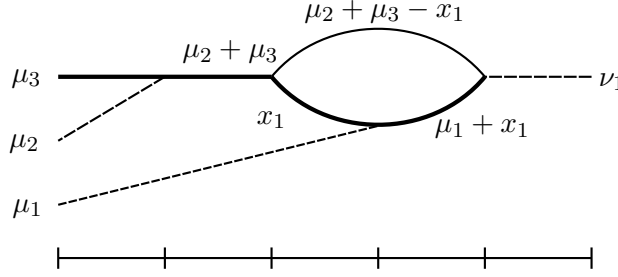


FIGURE 5. Mixed monodromy graph of genus 1.

5. CHAMBER BEHAVIOUR IN GENUS 0

In this section, we define a counting problem in the symmetric group generalising mixed Hurwitz numbers in genus 0. We use this to obtain recursive wall-crossing formulas in genus 0. As before, for fixed $\ell(\mu)$ and $\ell(\nu)$, let $m_0^k(C)$ be the polynomial expressing $M_0^k(\mu, \nu)$ in the chamber C . Moreover, let C_1 and C_2 be adjacent chambers separated by the wall $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$ (w.l.o.g $\delta > 0$ in C_1). We want to compute the wall-crossing for fixed $\ell(\mu)$ and $\ell(\nu)$

$$WC_{C_1}^{C_2}(\mu, \nu, k) = m_0^k(C_1) - m_0^k(C_2).$$

Remark 5.1. To study this wall-crossing problem, we define related Hurwitz-type counts generalising mixed Hurwitz numbers. It is an interesting feature of this Hurwitz-type counting problem that the wall-crossing induced by the piecewise polynomial structure can itself be expressed in of these Hurwitz numbers with smaller input data. For a precise formulation, see Definition 5.4 and Theorem 5.7.

We classify those reduced monodromy graphs Γ having different multiplicity in C_1 than in C_2 , since graphs with the same multiplicity cancel in $WC_{C_1}^{C_2}(0, \mu, \nu, k)$. By our discussion in Section 4.1, there are five cases of graphs contributing to $m_0^k(C_1)$, which contribute a different multiplicity $m_0^k(C_2)$:

- (1a) The graphs contributing to $m_0^k(C_1)$ (resp. $m_0^k(C_2)$) having an normal edge of weight δ (resp. $-\delta$) emerging from one of the first k vertices.
- (1b) The graphs contributing to $m_0^k(C_1)$ (resp. $m_0^k(C_2)$) having an non-normal edge of weight δ (resp. $-\delta$) emerging from one of the first k vertices.
- (1c) The graphs contributing to $m_0^k(C_1)$ (resp. $m_0^k(C_2)$) having an edge of weight δ (resp. $-\delta$) emerging from one of the last $r - k$ vertices.
- (2a) The graphs contributing to $m_0^k(C_1)$ (resp. $m_0^k(C_2)$) with a chain-path (see Definition 4.7) containing two edges e and e' (e coming before e' , such that $\omega(e) - \omega(e') = \delta$ (resp. $\omega(e) - \omega(e') = -\delta$).
- (2b) The graphs contributing to $m_0^k(C_1)$ (resp. $m_0^k(C_2)$) with a chain-path containing two edges e and e' (e coming before e'), such that $\omega(e) - \omega(e') = -\delta$ (resp. $\omega(e) - \omega(e') = \delta$).

Conditions (1a), (1b) and (1c) correspond to the fact that every edge weight must be greater than 0. Since $\delta < 0$ in C_2 , the graph Γ has multiplicity 0 in that chamber. Conditions (2a) and (2b) correspond to changes in the polynomials computing the counters for each chain path: The polynomial can change if $\mu_i - \omega(e)$ (e contained in a chain path starting at μ_i) or $\omega(e) - \omega(e')$ (e and e' contained in the same chain path) changes sign by crossing δ . Note that $\mu_i - \omega(e) = \omega(e') - \omega(e)$ for e' being the in-end μ_i . Thus, we obtaine the two cases (2a) and (2b).

The following idea will be our main tool in this section: We start with a mixed monodromy graph and cut it along some distinguished edge (resp. two distinguished edges). Since the graphs are of genus 0, we obtain two (resp. three) new mixed monodromy graphs. Our first step is classifying the pairs (resp. triples) of graphs we can obtain by this cutting process. Our second step is a regluing process. We will glue our graphs from two (resp. three) smaller graphs. The key observation here is, that in order to obtain a mixed monodromy graph again, this gluing has to respect the following:

- (1) The ordering of the chains of bold edges.
- (2) The monotonicity of the counters, if we glue edges to a new chain-path.

In order to formalise this, we introduce a new and more general Hurwitz-type counting problem, where these two conditions are framed in terms of so-called *start and end conditions*. In some sense, these start and end conditions remember the counter and position of the edge, which we cut in the first place. Analysing this regluing process, we obtain a recursive wall-crossing formula for this more general counting problem. To understand the general idea, we start by decomposing the graphs above into smaller graphs and thus make the mentioned cutting process more precise.

- (1a) Let Γ be a mixed monodromy graph of type $(0, \mu, \nu, k)$ with a normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, emanating from one of the first k vertices. We cut the graph along the edge δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), k_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, k_2)$ with $k_1 + k_2 = k$. Starting with two mixed monodromy graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), k_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, k_2)$, we want to glue them along the edge corresponding to δ . However, this does not always yield a mixed monodromy graph (e.g.: If δ is a normal edge in Γ_1 , but a bold edge in Γ_2 .) Thus, we need some compatibility condition for these graphs. In fact, in order to obtain a mixed monodromy graph of type $(0, \mu, \nu, k)$ with a normal edge of weight δ , the edge δ must be normal in Γ_1 and dashed in Γ_2 . Furthermore, if δ emanates from a chain of bold edges starting at μ_l in Γ_1 the edge δ must join with a chain of bold edges starting at μ_j in Γ_2 where $j > l$ (since the dashed and normal edges connect chains of bold edges). This corresponds to the end condition of type $(1, l, i)$ for Γ_1 and the start condition of type $(1, l)$ for Γ_2 in Definition 5.3, where i is the label we choose for the out-end of Γ_1 corresponding to δ .
- (1b) Let Γ be a mixed monodromy graph of type $(0, \mu, \nu, k)$ with a non-normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$. As before, we cut along δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), k_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, k_2)$. Gluing two graphs Γ_1 and Γ_2 of these types along δ , we see that in order to obtain a graph as in (1b), there are two types of conditions: Either δ is dashed in both Γ_1 and Γ_2 and if δ is contained in a chain path starting μ_l in Γ_1 , the in-end δ in Γ_2 must join with a chain of bold edges starting at μ_j in Γ_2 with $j > l$. Alternatively, δ is dashed in Γ_1 and bold in Γ_2 . Moreover if δ has counter c in Γ_1 , the first inner edge of the chain of bold edges starting at δ in Γ_2 must have counter $c' > c$. This corresponds to the end condition of type $(2, i, l, c)$ for Γ_1 and start condition of type $(2, l, c)$ in Definition 5.3, where i is the label we choose for the out-end of Γ_1 corresponding to δ .
- (1c) Let Γ be a mixed monodromy graph of type $(0, \mu, \nu, k)$ with a normal edge of weight $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$, emanating from one of the last $n - k$ vertices. As before, we cut along δ and obtain two graphs Γ_1 and Γ_2 of respective type $(0, \mu_I, (\nu_J, \delta), k_1)$ and $(0, (\mu_{I^c}, \delta), \nu_{J^c}, k_2)$. Here the only condition for the gluing process we require is δ only interacting with one of the last $n_j - k_j$ vertices in Γ_j ($j = 1, 2$). This corresponds to

the end and start condition $(3, i)$ in Definition 5.3, where i is the label we choose for the out-end of Γ_1 corresponding to δ .

- (2a) We once again start with a mixed monodromy graph of type $(0, \mu, \nu, k)$. We impose the condition that there is a chain path with two edges e_1, e_2 (e_1 appearing before e_2), such that

$$\omega(e_1) - \omega(e_2) = \delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j.$$

For

$$(9) \quad \omega(e_1) = \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j$$

and

$$(10) \quad \omega(e_2) = \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j,$$

and $\omega(e_1) - \omega(e_2) = \delta$ this translates to $I_2 = I_1 \sqcup I^c$ and $J_2 = J_1 \sqcup J^c$. Cutting along the edges e_1 and e_2 , we obtain three graphs Γ_1, Γ_2 and Γ_3 of respective types

$$(11) \quad (0, \mu_{I_1}, \left(\nu_{J_1}, \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j \right), k_1),$$

$$(12) \quad (0, \left(\mu_{I^c}, \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j \right), \left(\nu_{J^c}, \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j \right), k_2),$$

$$(13) \quad (0, \left(\mu_{I_2^c}, \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j \right), \nu_{J_2^c}, k_3),$$

where $k_1 + k_2 + k_3 = k$. Regluing graphs of these respective types correspond to the gluing process in (1b). Thus we need an end condition of type $(2, l, c, i)$ for Γ_1 , start condition of type $(2, l, i)$ for Γ_2 , end condition of type $(2, l, c, i)$ for Γ_2 and start condition of type $(2, l, i)$ for Γ_3 . (If $I_1 = p$ and $J_1 = \emptyset$, we only cut at $\delta + \mu_p$ and thus obtain only Γ_2 and Γ_3 .)

- (2b) Starting with a mixed monodromy graph of type $(0, \mu, \nu, k)$, with a chain path containing two edges e_1 and e_2 (with e_1 appearing before e_2 , such that

$$\omega(e_1) - \omega(e_2) = -\delta = \sum_{i \in I^c} \mu_i - \sum_{j \in J^c} \nu_j.$$

For $\omega(e_1)$ and $\omega(e_2)$ as in Equation (9) and (10) respectively, and $\omega(e_1) - \omega(e_2) = \delta$ this translates to $I_2 = I_1 \sqcup I$ and $J_2 = J_1 \sqcup J$. Similar as in (2a), we cut along e_1 and e_2 to obtain three graphs Γ_1, Γ_2 and Γ_3 of respective types

$$(0, \mu_{I_1}, \left(\nu_{J_1}, \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j \right), k_1),$$

$$(0, \left(\mu_I, \sum_{i \in I_1} \mu_i - \sum_{j \in J_1} \nu_j \right), \left(\nu_J, \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j \right), k_2),$$

$$(0, \left(\mu_{I_2^c}, \sum_{i \in I_2} \mu_i - \sum_{j \in J_2} \nu_j \right), \nu_{J_2^c}, k_3),$$

where $k_1 + k_2 + k_3 = k$. Regluing graphs of these respective types, we need to impose the same end and start conditions as in (2a).

Notation 5.2. We fix two partitions μ and ν .

- (1) For a subset $I = \{i_1, \dots, i_n\}$ (where $i_j < i_{j+1}$) of $\{1, \dots, \ell(\mu)\}$ and positive integers δ, j ($j \notin I$), we denote by $(\mu_I, \delta)_j$ the partition $(\mu_{i_1}, \dots, \mu_{i_j}, \delta, \mu_{i_j+1}, \dots, \mu_{i_n})$.
- (2) Let $(\sigma_1, \tau_1, \dots, \tau_r, \sigma_2)$ be a mixed factorisation with $\mathcal{C}(\sigma_1) = \mu$ and $\mathcal{C}(\sigma_2) = \nu$. We define $\tau^1(l)$ to be the transposition with the biggest position containing elements of the cycle of σ_2 labeled l . Moreover, let $t^1(l)$ be the position of $\tau^1(l)$.
- (3) Let $(\sigma_1, \tau_1, \dots, \tau_r, \sigma_2)$ be a mixed factorisation with $\mathcal{C}(\sigma_1) = \mu$ and $\mathcal{C}(\sigma_2) = \nu$. We define $\tau^2(l)$ to be the transposition with the smallest position containing elements of the cycle of σ_1 labeled l . Moreover, let $t^2(l)$ to be the position of $\tau^2(l)$.

Definition 5.3. Let μ, ν, k be data as before. Let $\eta = (\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ be a mixed factorisation of type $(0, \mu, \nu, k)$, such that $\tau_i = (r_i \ s_i)$.

We begin by defining end-conditions:

- (1) We say η satisfies end condition $(1, l, i)$ if
 - $t^1(i) \leq k$
 - $\sum_{j=1}^{l-1} \mu_j + 1 \leq s_{t^1(i)} \leq \sum_{k=1}^l \mu_k$

In monodromy graph language, this corresponds to the following picture: The out-end corresponding to ν_i is coloured normal and emanates from the chain of bold edges starting at μ_l (see Figure 6).

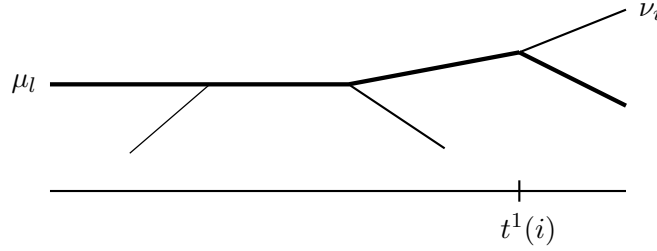
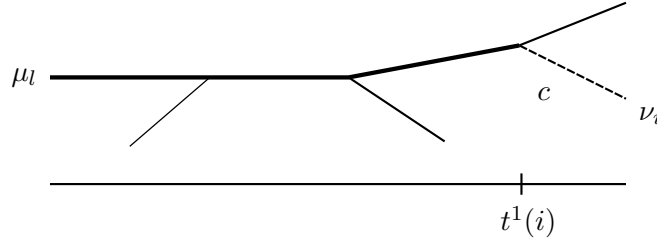


FIGURE 6. Schematic drawing of end condition $(1, l, i)$.

- (2) We say η satisfies end condition $(2, l, c, i)$ if
 - $t^1(i) \leq k$
 - $s_{t^1(i)} = \sum_{j=1}^{l-1} \mu_j + c$ for $0 \leq c < \mu_l$

In monodromy graph language, this corresponds to the following picture: The out-end corresponding to ν_i is coloured dashed, has counter c and emanates from the chain of bold edges starting at μ_l . (See Figure 7).

- (3) We say η satisfies end condition $(3, i)$ if
 - $t^1(i) > k$

FIGURE 7. Schematic drawing of end condition $(2, l, c, i)$.

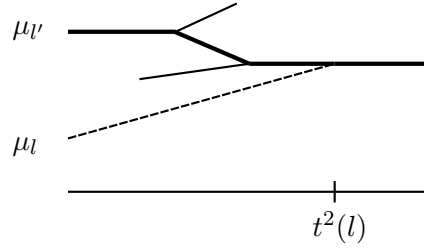
In monodromy graph language, this corresponds to the following picture: The out-end corresponding to ν_i emanates from a vertex whose position is greater than k .

Now we define start conditions.

(1) We say η satisfies start condition $(1, l)$ if

- $t^2(l) \leq k$
- $s_{t^2(l)} \geq \sum_{j=1}^l \mu_j + 1$

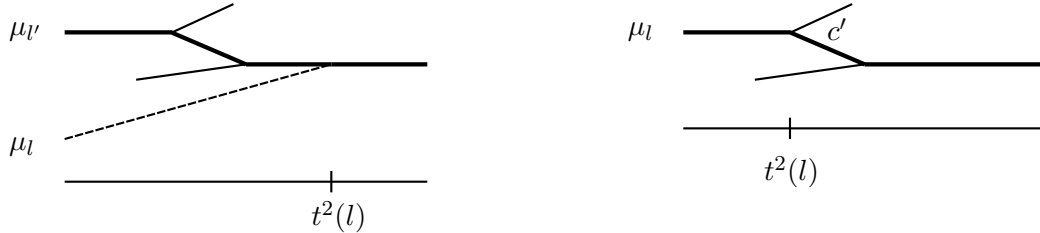
In monodromy graph language, this corresponds to the following picture: The in-end corresponding to μ_l is coloured dashed and joined with a chain of bold edges emanating at $\mu_{l'}$ for $l' > l$. (See Figure 8).

FIGURE 8. Schematic drawing of end condition $(1, l)$ for $l' > l$.

(2) We say η satisfies start condition $(2, l, c)$ if

- $t^2(l) \leq k$
- $s_{t^2(l)} \geq \sum_{j=1}^{l-1} \mu_j + c$ for $0 \leq c < \mu_l$

In monodromy graph language, this corresponds to one of the following two pictures: Either μ_l is joined with a chain of bold edges emanating at μ_m for $m > l$ or there is a chain of bold edges starting at μ_l and the first counter is greater or equal to c .

FIGURE 9. Schematic drawing of the two possible graphs for end condition $(2, l, c)$: $l' > l$ in the left graph; $c' \geq c$ in the right graph.

(3) We say η satisfies start condition $(3, l)$ if

- $t^2(l) > k$

In monodromy graph language, this corresponds to the following picture: The in-end corresponding to μ_l is adjacent to a vertex whose position is greater than k .

Definition 5.4. Let S be a set of start conditions and E a set of end conditions, i.e. for each $i \in \ell(\mu)$ (resp. $j \in \ell(\nu)$), there exists at most one tuple $(1, l)$, $(2, l, c)$ or $(3, l)$ (resp. $(1, l, i)$, $(2, l, c, i)$ or $(3, i)$) for some $l \in \{1, \dots, \ell(\mu)\}$, $c \in \{0, \dots, \mu_l - 1\}$ (resp. $i \in \{1, \dots, \ell(\nu)\}$, $l \in \{1, \dots, \ell(\mu)\}$, $c \in \{0, \dots, \mu_l - 1\}$), which is contained in S , resp. E . Then we define $M_g^k(\mu, \nu, S, E)$ to be the number of all mixed factorisations $(\sigma_1, \tau_1, \dots, \tau_m, \sigma_2)$ of type $(0, \mu, \nu, k)$ with σ_1 as in Equation (2) satisfying the conditions in E and S .

Remark 5.5. For $E = S = \emptyset$, we obtain mixed Hurwitz numbers. Moreover, our methods from subsection 4.1 can be applied to this generalised version to obtain piecewise polynomiality in the entries of μ and ν and the information in S and E with chambers given by \mathcal{W} .

By the same arguments as in Section 4.1, $M_0^k(\mu, \nu, S, E)$ may be expressed as a polynomial in the entries of μ and ν in each chamber induced by the hyperplane arrangement \mathcal{W} . We denote the polynomial in the chamber C by $m_0^k(\mu, \nu, S, E)(C)$.

Before we are ready to state the main theorem of this section, we introduce some notation.

Notation 5.6. Let μ be an ordered partition, $i \in \{1, \dots, \ell(\mu)\}$ and δ an integer, then we define the partition $(\mu, \delta)_i = (\mu_1, \dots, \mu_{i-1}, \delta, \mu_i, \dots, \mu_{\ell(\mu)})$.

Moreover, let S be a set of start conditions and $I \subset \{1, \dots, \ell(\mu)\}$, then S_I is the set of all start condition $(1, l)$, $(2, l, c)$ oder $(3, l)$ with $l \in I$.



FIGURE 10. The case (1a) and (2a) simultaneous on the left, the case (1b) and (2a) simultaneous on the right.

Theorem 5.7. Let μ and ν be ordered partitions, k a non-negative integer and C_1, C_2 chamber separated by the wall defined by $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$ for subsets $I \subset \{1, \dots, \ell(\mu)\}$, $J \subset \{1, \dots, \ell(\nu)\}$. Then

$$W_{C_1}^{C_2}(\mu, \nu, k) = m_0^k(\mu, \nu, S, E)(C_1) - m_0^k(\mu, \nu, S, E)(C_2)$$

is a sum of products with factors of δ and two or three polynomials $m_0^{k'}(\mu', \nu', S', E')(C_1)$ (resp. $m_0^{k'}(\mu', \nu', S', E')(C_2)$), where the apostrophes indicate smaller input data. More precisely, the data k', μ', ν', S', E' has to satisfy $k' < k$, $\mu' = (\mu_I, \delta)_i$ (for some $i \in I$), $\nu' = (\nu_J, \delta)_j$ (for some $j \in J$), S' is the union of S_I and a start condition corresponding to the entry δ and E' the union of E_J and an end condition corresponding to the entry δ for $m_0^{k'}(\mu', \nu', S', E')(C_1)$ (resp. replacing δ by $-\delta$ for $m_0^{k'}(\mu', \nu', S', E')(C_2)$).

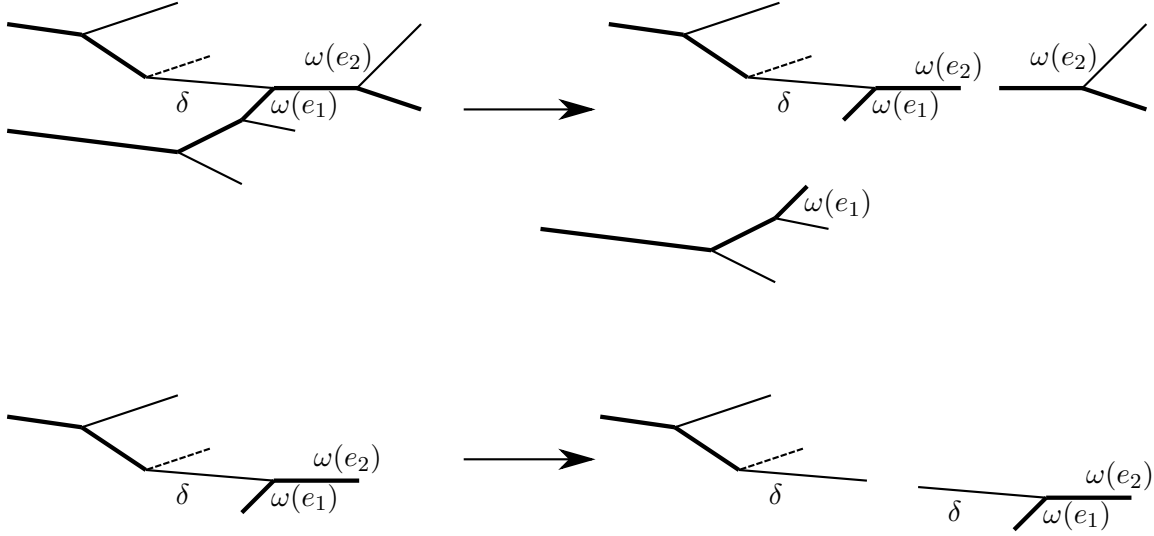


FIGURE 11. Schematic drawing of the cut-and-join process corresponding to cases (1a) and (2a) simultaneously.

Proof of Theorem 5.7. By the discussion at the beginning of this section, we already have an interpretation of the cutting and gluing process in terms of the polynomials $m_0^{k'}(\mu', \nu', S', E')(C_1)$ and $m_0^{k'}(\mu', \nu', S', E')(C_2)$. The proof is straightforward but involves many cases (in fact the complete formula for $W_{C_1}^{C_2}(\mu, \nu, k)$ in terms of the $m_0^{k'}(\mu', \nu', E', S')(C_i)$ covers more than three pages). We will work out the formula for (2a), since this is the most difficult case. The remaining cases can be worked out analogously.

As discussed before, we identify those graphs with different multiplicity in C_1 than in C_2 . The following is the number of all graphs as in (2a), where e_1 (see Equation (9)) is not an edge adjacent to an in-end (and analogously in chamber C_2):

$$\sum_{\substack{I_1 \subset I^c, J_1 \subset J^c \\ |I_1| > 1 \text{ or } J_1 \neq \emptyset}} \sum_{k_1 + k_2 + k_3 = k} \binom{m - k}{|I_1| + |J_1| - 1 - k_1, |I^c| + |J^c| - k_2, |I - I_1| + |J - J_1| - 1 - k_3} \\ \sum_{t \in I_1} \sum_{1 \leq l \leq l' \leq \mu_t} \left(m_0^{k_1}(\mu_{I_1}, (\nu_{J_1}, \omega(e_1)), E_{J_1} \cup \{(2, t, l, |J_1| + 1)\}, S_{J_1})(C_1) \cdot \right. \\ m_0^{k_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2)), E_J \cup \{(2, t, l', |J| + 1)\}, S_I \cup \{(2, t, l)\})(C_1) \cdot \\ \left. m_0^{k_3}((\mu_{I^c - I_1}, \omega(e_2))_t, \nu_{J^c - J_1}, E_{I^c - I_1}, S_{J^c - J_1} \cup \{(2, t, l')\})(C_1) \right),$$

where $\omega(e_2) = \omega(e_1) + \delta$. As mentioned before, by cutting along the two distinguished edges, we obtain three graphs of respective types as in Equation (11), (12) and (13). Each of these types corresponds to one the three factors. The binomial coefficient counts the number of possible orderings on the vertices not affected by the monotonicity condition. All the other cases work similar, however what needs to be checked is that we obtain every graph exactly once. In fact, by the method above, we overcount, since (1a) and (2a) or (1b) and (2a) may appear simultaneously, which corresponds to the local picture illustrated in Figure 10: The same graph Γ can be reglued from pieces that we obtain in case (1a) and (2a). By

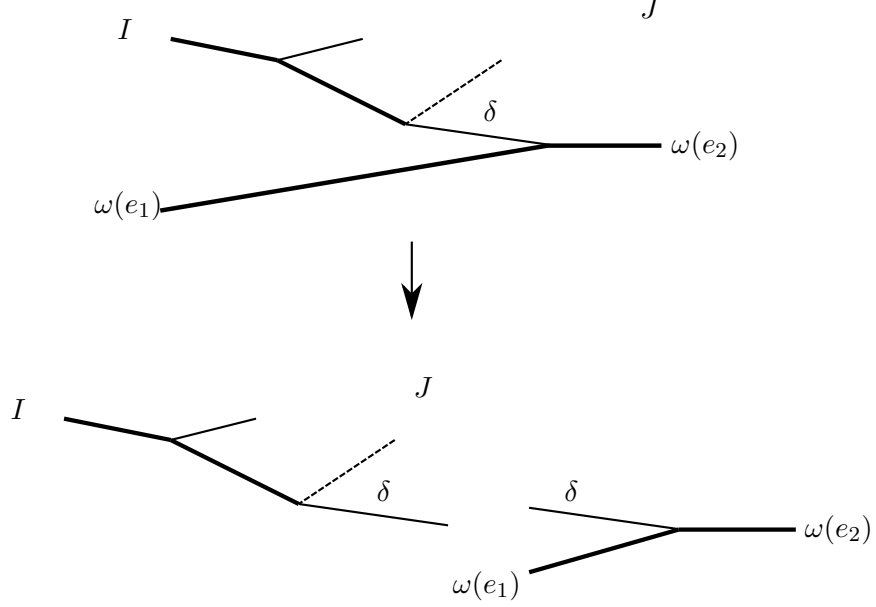


FIGURE 12. Upper graph: Schematic drawing the graph corresponding to the correction term with set of in-ends indexed by $I \cup \{e_1\}$ and set of out-ends indexed by $J \cup \{e_2\}$. Lower graph: Schematic drawing of the graph obtained by cutting along δ .

cutting along δ we obtain case (1a), however cutting along e_1 and e_2 , we obtain case (2a). This cut-and-join process is illustrated in Figure 11. The upper picture is a schematic drawing of the cutting along the edges e_1 and e_2 , when case (1a) and (2a) happen at the same time. To compute the correction term, we need to remove all graphs G as shown in left of the lower picture. This is done by cutting along δ counting all graphs \tilde{G} with out-end δ as in the right hand side of the lower picture and realising that the multiplicity of G is δ times the multiplicity of \tilde{G} . In terms of the formula this means that we need to subtract the number of graphs as in upper picture in Figure 12 from the factor

$$m_0^{k_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2)), E_J \cup \{(2, t, l', |J| + 1)\}, S_I \cup \{(2, t, l)\})(C_1)$$

in the product. This is done by cutting along δ , counting all graphs as in the left of the lower picture in Figure 12 and realising that regluing the graph yields a factor of δ . Thus, we need to subtract the following term from

$$m_0^{k_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2)), E_J \cup \{(2, t, l', |J| + 1)\}, S_I \cup \{(2, t, l)\})(C_1)$$

in each summand:

$$\sum_{\substack{h \in I: \\ h < t}} \delta \cdot m_0^{k_2-1}(\mu_I, (\nu_J, \delta), E_J \cup (1, h, |J| + 1), S_I).$$

By a similar argument, we see that when the cases (1b) and (2a) appear simultaneously, we need to subtract the following term from

$$m_0^{k_2}((\mu_I, \omega(e_1))_t, (\nu_J, \omega(e_2)), E_J \cup \{(2, t, l', |J| + 1)\}, S_I \cup \{(2, t, l)\})(C_1)$$

in each summand:

$$\sum_{\substack{h \in I: \\ h < t}} \sum_{g=1}^{\mu_h} \delta \cdot m_0^{k_2-1}(\mu_I, (\nu_J, \delta), E_J \cup (2, h, g, |J| + 1), S_I).$$

□

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